Towards an Encompassing Theory of Network Models: Reply to Brusco, Steinley, Hoffman, Davis-Stober, & Wasserman

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Abstract

Network models like the Ising model are increasingly used in psychological research. In a recent article published in this journal, Brusco, Steinley, Hoffman, Davis-Stober, and Wasserman (in press) provide a critical assessment of the conditions that underlie the Ising model and the eLasso method that is commonly used to estimate it. In this commentary, we show that their main criticisms are unfounded. First, where Brusco et al. suggest that Ising models have little to do with classical network models such as random graphs, we show that they can be fruitfully connected. Second, if one makes this connection it is immediately evident that Brusco et al.'s second criticism – that the Ising model requires complete population homogeneity and does not allow for individual differences in network structure – is incorrect. In particular, we establish that if every individual has their own topology, and these individual differences instantiate a random graph model, the Ising model will hold in the population. Hence, population homogeneity is sufficient for the Ising model, but it is not necessary, as Brusco et al. suggest. Third, we address Brusco et al.'s criticism regarding the sparsity assumption that is made in common uses of the Ising model. We show that this criticism is misdirected, as it targets a particular estimation algorithm for the Ising model rather than the model itself. We also describe various established and validated approaches for estimating the Ising model for networks that violate the sparsity assumption. Finally, we outline important avenues for future research.

Keywords: Ising model, random-cluster model, graphical model, random graph model, network model, network psychometrics, homogeneity

Towards an Encompassing Theory of Network Models: Reply to Brusco, Steinley, Hoffman, Davis-Stober, & Wasserman

Introduction

In a recent article that was published in this journal, Brusco, Steinley, Hoffman, Davis-Stober, and Wasserman (in press, henceforth BSHDW) provided a critical assessment of the conditions that underlie the Ising model and the eLasso method that is commonly used in psychology to estimate the Ising model. Their reason for this critical assessment was that "*little concern has been reflected in the psychopathology literature with respect to the data assumptions and conditions that underlie the Ising (1925) model and/or the methods designed for its estimation*" (p. 13). BSHDW identified two critical assumptions of the Ising model and/or the eLasso method that they believe are unjustified in applications of psychopathology, or psychological science more generally. One of these key assumptions is that the population is homogeneous with respect to the model's characteristics, i.e., that persons are i.i.d. replications of the same associative structure. The other key assumption is the assumption of the eLasso method that the underlying associative structure is sparse in nature, i.e., that there exist relatively few connections between observed variables. The authors then conclude that there are "*serious concerns regarding the implementation of eLasso in psychopathological research*" (p. 1) and propose three distinct approaches that may help mitigate these concerns.

An apparently innocuous critique permeates the authors' assessment: "*Although some researchers will maintain that the debate over the terms networks vs. dependence graph is merely one of semantics, we believe there is a salient distinction*" (p. 2). Whereas we believe that network models comprise both graphical- and random graph models, BSHDW seem to reserve the term network model exclusively for random graph models. This suggests that it would be wrong to refer to graphical models such as the Ising model as a network model, and by extension it would also be wrong to call our area of research *network psychometrics*. We respectfully disagree with this view.[1] We do agree, however, that there is a clear, formal distinction between the class of random graph models and the class of graphical models. That the two frameworks span different areas of the statistical literature emphasizes their distinction. But even though it may appear that these model families have little in common, an idea that underlies BSHDWs' critiques and proposals, we will use the theory of Fortuin and Kasteleyn (1972) to argue that these models are not only closely connected, but that further study of this connection in fact defines an absolutely crucial avenue for research in network psychometrics.

The theory of Fortuin and Kasteleyn relates the graphical model to the random graph model by showing that both models form the marginals of an encompassing network model. This encompassing network theory turns out to be quite useful in understanding BSHDWs' assessment of the Ising model and the solutions BSHDW propose to mitigate raised concerns. As an example, we will consider the authors' assessment regarding homogeneity. We agree with BSHDW that homogeneity of the population with respect to the Ising model's characteristics will guarantee the model's fit to cross-sectional data. In this commentary, however, we wish to address the mistaken belief that it is the only way to ensure the model's fit. In fact, as we will show here, homogeneity is not necessary, as the authors argue, but merely sufficient. Other sufficient conditions exist however, and we will argue that studying these can reveal important connections between random graph models and graphical models. We will illustrate a particular alternative that does not require that the population is homogeneous with respect to the model's characteristics. Using a recent idiographic network characterization of the Ising model (Marsman, 2019; Savi, Marsman, van der Maas, & Maris, 2019) that builds on the theory of Fortuin and Kasteleyn (1972) we demonstrate how network structures that are unique to the individual can still generate

[1] Observe that this not only chastises network psychometrics for using the network name, but also other fields that centre around graphical models, such as neural networks (Hopfield, 1982) and Bayesian networks (Pearl, 1988).

an Ising model cross-sectionally. The theory of Fortuin and Kasteleyn furthermore reveals interesting connections between two of BSHDWs' proposed solutions, the (stochastic) block model and a low-rank approximation to the gram matrix, and existing psychometric modeling approaches.

The remainder of this commentary is organized as follows. First, we introduce the two types of network models, the random graph model and the graphical model, and discuss how the theory of Fortuin and Kasteleyn relates the two statistical frameworks. Second, we use Fortuin and Kasteleyn's theory to address the mistaken belief that a homogeneous population is required for the cross-sectional application of Ising models. Third, we address some omissions, and discuss several results and models that BSHDW have missed in their discussion centred around the sparsity assumption.

An Encompassing Network Theory: First Steps

We first introduce some notation. Let the vector $\mathbf{x} = [x_1,\, x_2,\, \dots,\, x_p]^\mathsf{T}$ denote the states of p nodes in a p-variable graph or network. The vector $\mathbf{w} = [w_{11},\, w_{12},\, \dots w_{23},\, \dots w_{(p-1)p}]^\mathsf{T}$ is used to denote the $\binom{p}{2}$ network relations, with $w_{ij} \in \{0,\, 1\}$, where $w_{ij} = 1$ implies that nodes i and j are directly connected and $w_{ij} = 0$ implies that these nodes are not directly connected.

A random graph model is characterized by a particular probability distribution over topological structures, i.e., it is a model for the different configurations of connections between nodes of the graph or network: $p(\mathbf{W} = \mathbf{w})$. One example could be the modeling of the connections between neurons in a person's brain, where the neurons are represented as the nodes of a network $\mathbf{x}$ and the dendrite connections then represent the edges of the network $\mathbf{w}$. A well-known statistical model that falls in the class of random-graph models is the model of Erdős and Rényi (1960)[2]

$$p(\mathbf{W} = \mathbf{w}) = \prod_{i=1}^{p-1} \prod_{j=i+1}^{p} \theta_{ij}^{w_{ij}}\, (1 - \theta_{ij})^{1-w_{ij}},$$

[2] This model is often referred to as *the* random graph model. Here, however, we will refer to it as the Erdös–Rényi model.

where the w_{ij} are independent Bernoulli(θ_{ij}) variables. Observe that random graph models describe the relational patterns that may exist between the network's nodes but that they do not model the network's node states $\mathbf{x}$. Thus, the random graph model describes the way the edges in a graph are distributed over pairs of nodes, without saying anything about the nodes.

In contrast, graphical models do describe the patterns in the network's node states, such as the activity patterns of the neurons in our brain network example: $p(\mathbf{X} = \mathbf{x})$. Observe that the graphical model does not model the connections between nodes but treats them as fixed entities. A prime example of such a graphical model is the Ising model,

$$p(\mathbf{X} = \mathbf{x}) = \frac{\exp\left(\sum_{i=1}^{p} x_i \mu_i \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} x_i x_j \sigma_{ij}\right)}{\sum_{\mathbf{x}} \exp\left(\sum_{i=1}^{p} x_i \mu_i \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} x_i x_j \sigma_{ij}\right)} \tag{1}$$

where $x_i \in \{-1, \ +1\}$, for $i = 1, \ \ldots, \ p$, and the sum in the denominator is with respect to all 2^p possible states of the vector $\mathbf{x}$. The Ising model comprises two parameters (see Marsman et al., 2018, for details), the vector of main effects or activation thresholds $\boldsymbol{\mu}$ and the matrix $\boldsymbol{\Sigma} = [\sigma_{ij}]$ of pairwise interactions. Thus, graphical models such as the Ising model describe the probability distribution of the node states as a function of the network structure that is encoded in the edge weights σ_{ij}.

The random graph model and the graph model are thus complementary to each other, as they model two distinct aspects of the network. In the view of BSHDW, however, interpreting the Ising model as a network "*can be problematic when researchers seek to apply standard measures of social network theory, such as centrality, to the edge weights, which may not be appropriate*" (p. 2). We agree with BSHDW that we should carefully review new methods, and also agree that there is no guarantee that an interpretation of measures developed for random graph models seamlessly carries over to applications of graphical models. Indeed, an Erdös–Rényi model seems to be a poor null-model for centrality measures defined on the connectivity matrix of graphical models such as the Ising model. However, we also think that BSHDW have an exclusive focus on how the two statistical modeling frameworks differ. We would suggest a change in direction and advocate a focus on what we can learn from the two modeling approaches and the relations between them.

This is what two physicists, Cees Fortuin and Piet Kasteleyn, did almost fifty years ago. In physics, graphical models such as the Ising model are used to model the behavior of particles (e.g., magnetism), and random graph models such as the Erdös–Rényi model are used to model the way that particles may trickle through a porous object (e.g., percolation; Broadbent & Hammersley, 1957). Fortuin and Kasteleyn observed several unexplained relations between the phenomena that were studied in these two disparate areas of research and set out to discover if this was more than pure coincidence (see the appendix of Grimmet, 2006). In a series of papers, Fortuin and Kasteleyn worked out how the two models relate (Fortuin, 1972a, 1972b; Fortuin & Kasteleyn, 1972) by formulating an encompassing network model that describes both aspects of the network, i.e., the states of the variables in the network and the connections between them,

$$p(\mathbf{x},\, \mathbf{w}) = p(\mathbf{x} \mid \mathbf{w})\, p(\mathbf{w}) = p(\mathbf{w} \mid \mathbf{x})\, p(\mathbf{x}). \tag{2}$$

They then showed that the Ising model in Eq. (1) is the node-marginal of the encompassing network model, i.e.,

$$p(\mathbf{X} = \mathbf{x}) = \sum_{\mathbf{w}} p(\mathbf{X} = \mathbf{x} \mid \mathbf{W} = \mathbf{w})\, p(\mathbf{W} = \mathbf{w}), \tag{3}$$

where the sum is with respect to all $2^{\binom{p}{2}}$ possible topological states $\mathbf{w}$. Observe that this characterization of the Ising model interprets the network's topology —the set of connections between variables in the network— as a random effect. The edge-marginal of the encompassing network model,

$$p(\mathbf{W} = \mathbf{w}) = \sum_{\mathbf{x}} p(\mathbf{W} = \mathbf{w} \mid \mathbf{X} = \mathbf{x})\, p(\mathbf{X} = \mathbf{x}),$$

where the sum is with respect to all 2^p possible node states $\mathbf{x}$, is a random graph model that describes the network's topological states. In the ensuing sections, we show that this encompassing network theory sheds new light on BSHDW's contributions. But, before we reanalyze BSHDW's contributions, we first turn to the different marginal, joint, and conditional models in Eq. (2).

Fortuin and Kasteleyn's encompassing network model (Eq. (6) in the appendix) builds on two deterministic rules:

(1) If two variables are connected then they must be in the same state: $(W_{ij} = 1) \implies (X_i = X_j)$.

(2) If two variables are in different states then they must be disconnected: $(X_i \neq X_j) \implies (W_{ij} = 0)$.

These causal rules allowed Fortuin and Kasteleyn to relate the behavior of nodes to the behavior of edges in their encompassing network model,

$$p(\mathbf{X} = \mathbf{x},\, \mathbf{W} = \mathbf{w}) \propto \prod_{i=1}^{p-1} \prod_{j=i+1}^{p} \left\{\theta_{ij}\, \delta_{(w_{ij},1)}\, \delta_{(x_i,\, x_j)} + (1 - \theta_{ij})\, \delta_{(w_{ij},0)}\right\} \exp\left(\sum_{i=1}^{p} x_i \mu_i\right),$$

where $\delta_{(a,\, b)}$ is an indicator function that is equal to one whenever $a = b$ and zero otherwise. The appendix confirms that this encompassing network model generates the Ising model in Eq. (1) after averaging over the topological structures. The appendix also confirms that averaging over the variable states $\mathbf{x}$ leads to a random graph model known as the random-cluster model,

$$p(\mathbf{W} = \mathbf{w}) = \frac{\prod_{i=1}^{p-1} \prod_{j=i+1}^{p} \theta_{ij}^{w_{ij}} (1 - \theta_{ij})^{1 - w_{ij}} \prod_{c=1}^{\kappa(\mathbf{w})} \beta_c}{\sum_{\mathbf{w}} \prod_{i=1}^{p-1} \prod_{j=i+1}^{p} \theta_{ij}^{w_{ij}} (1 - \theta_{ij})^{1 - w_{ij}} \prod_{c=1}^{\kappa(\mathbf{w})} \beta_c},$$

where $\kappa(\mathbf{w})$ denotes the number of connected components or open clusters[3] in the topology $\mathbf{w}$. A random-cluster model with clustering coefficients $\beta_c > 1$ favors network structures with more open clusters over structures with fewer open clusters, and it is the opposite case for clustering coefficients $0 < \beta_c < 1$. The Erdös–Rényi model is the special case of a random-cluster model with unit clustering weight, i.e., $\prod_c^{\kappa(\mathbf{w})} \beta_c = \lambda^{\kappa(\mathbf{w})}$, with $\lambda = 1$. Finally, the latent topology model $p(\mathbf{x} \mid \mathbf{w})$ is used to assign values to the nodes $\mathbf{x}$ of the topology $\mathbf{w}$ —it "colors" the graph— and randomly assigns the same value to each variable in a cluster, independently for each cluster.

The seminal work of Fortuin and Kasteleyn shows that the Ising model and the random-cluster model are the marginals of an encompassing network model, which

[3] An open cluster is a group of nodes that lie on an open path together, so that one can travel along all of the nodes in the group using the edges in $\mathbf{w}$.

suggests that phenomena that are observed in random graph models can mirror the phenomena that are observed in graphical models and vice versa. The following result underscores this idea and relates the concepts of correlation and connectivity (Grimmet, 2006, Theorem 1.16)

$$p(X_i = X_j) = \frac{1}{2} + \frac{1}{2}p(X_i \leftrightarrow X_j) \geq \frac{1}{2}, \tag{4}$$

which expresses the marginal probability that the two variables are in the same state as a function of the probability that the two variables are directly or indirectly connected.[4] One particular consequence of this relation is that variables are non-negatively correlated in the population (Fortuin, Kasteleyn, & Ginibre, 1971), which is a well-established psychometric phenomenon known as the positive manifold (Savi et al., 2019). The idea that the two models mirror the same underlying phenomena drove Fortuin and Kasteleyn to formulate their theory and allowed them to uncover the relationship between phenomena observed in models of percolation and magnetism. In this paper, we will use the theory to analyze BSHDW's concerns and proposed solutions.

Before we reanalyze BSHDW's contributions, however, we wish to drive home the message that the two fields of network models are intimately connected, and point out that there are two alternative theories that merge graph models and random-graph models; the divide and color model of Häggström (2001) and the random triangle model of Jonasson (1997). The two approaches formulate different encompassing network models, and consequently, inspire different marginals. The divide and color model is of the form of Eq. (3) but replaces the random-cluster model with an Erdös–Rényi model. Even though there is no closed-form expression for the marginal $p(\mathbf{x})$ of this encompassing model, and hence it does not generate an Ising model, several of its properties are known (Bálint, 2010; Häggström, 2001), and are closely related to that of other latent variable characterizations of the Ising model (e.g., Fortuin et al., 1971;

[4] This relation holds for every marginal $p(\mathbf{w})$, i.e., every random graph model, when combined with the aforementioned latent topology model $p(\mathbf{x} \mid \mathbf{w})$ to color the graph, but only considers the case sans main effects μ. Eq. (1.1) in Steif and Tykesson (2017) and Theorem 2 in Cioletti and Vila (2016) offer generalizations for cases including main effects.

Holland & Rosenbaum, 1986). The random-triangle model of Jonasson (1997) offers yet another approach, in which the focus on clusters in the latent topology representation of Eq. (3) is abandoned in favor of triangles. This representation can, in principle, also characterize an Ising model (e.g., Häggström & Jonasson, 1999). In sum, there exist several interesting connections between random graph models and graphical models, which confirms that their distinction may be less salient than BSHDW make us believe.

Stochastic Block Models Generate Low-Rank Ising and Bi-Factor IRT Models

As a testament to the above sentiment, we consider the stochastic block model of Holland, Laskey, and Leinhardt (1983) —a random-graph model with community structure (Karrer & Newman, 2011)— that was proposed by BSHDW to block or factor the network's nodes. The original stochastic block model formulation stipulates the community structure on an Erdös–Rényi model. Here, we follow Savi et al. (2019) and impose the community structure on the random-cluster model. The reason is that for the latter set-up, the marginal $p(\mathbf{x})$ is analytically available, whereas it is not for the former set-up. Eq. (4) ensures that the cross-sectional patterns of both set-ups will be similar.

In its simplest form, the stochastic block model differentiates between the probability θ_W of laying an edge between nodes within the same community, and the probability θ_B of laying an edge between nodes of different communities:

$$p(\mathbf{W} = \mathbf{w}) = \frac{\prod_{i,j \in E_W} \theta_W^{w_{ij}} (1-\theta_W)^{1-w_{ij}} \prod_{i,j \in E_B} \theta_B^{w_{ij}} (1-\theta_B)^{1-w_{ij}} \prod_{c=1}^{\kappa(\mathbf{w})} \beta_c}{\sum_{\mathbf{w}} \prod_{i,j \in E_W} \theta_W^{w_{ij}} (1-\theta_W)^{1-w_{ij}} \prod_{i,j \in E_B} \theta_B^{w_{ij}} (1-\theta_B)^{1-w_{ij}} \prod_{c=1}^{\kappa(\mathbf{w})} \beta_c},$$

where E_W comprises the node pairs that are in the same community and E_B the node pairs that are in different communities. In Savi et al. (2019) it was shown that this random-cluster model generates an Ising model with a community structure in its association matrix

$$\mathbf{\Sigma} = [\sigma_{ij}] = \sigma_B \mathbf{1}_n + (\sigma_W - \sigma_B)\mathbf{C},$$

where $\mathbf{C}$ is a symmetric $p \times p$ matrix with elements c_{ij} that are equal to one whenever i and j are in the same community and zero otherwise. This correspondence follows from the (monotonic) relation between the Ising model's associations and the edge probabilities in the random-cluster model (Fortuin & Kasteleyn, 1972; Grimmet, 2006)

$$\sigma_{ij} = -\frac{1}{2}\log(1-\theta_{ij}). \tag{5}$$

If $\theta_B < \theta_W$, which implies that there are more connections between members of the same community than between members of different communities, the associations in the related Ising model have the same ordering, $\sigma_B < \sigma_W$, so that all elements of the association matrix $\mathbf{\Sigma}$ are positive and the related Ising model is consistent with a bi-factor item response theory (IRT) model (Marsman, Maris, Bechger, & Glas, 2015; Savi et al., 2019, p. 1056–1057).

The theory of Fortuin and Kasteleyn thus reveals that the proposal of BSHDW to use a block modeling approach to factor nodes is strongly related to existing network psychometric approaches. Specifically, their approach is strongly related to the low-rank approach pioneered in Marsman et al. (2015), which stipulates a low-rank approximation to the Ising model's association matrix, and which will lead to the community structure formulated above. BSHDW's suggestion to block both nodes and subjects will then generate mixtures of such low-rank Ising models, or, equivalently, mixtures of IRT models (Marsman et al., 2018, 2015), each with a potentially different bi-factor structure. That we were able to connect BSHDW's random-graph inspired proposal to existing psychometric models underscores the value of Fortuin and Kasteleyn's theory.

An Idiographic Interpretation of Cross-Sectional Ising Models

One of the primary critiques of BSHDW against the Ising model concerns the assumption that individuals are exchangeable, i.e., that they are i.i.d. replications of a single associative structure. But in their words, "*there is little compelling evidence that the Ising model is well-suited for identifying good graphical models of psychopathological data*" (p. 8). One reason for this is that "*psychopathology data may not consist of n i.i.d. realizations from the Ising model*" (p. 8). We would argue that this assumed heterogeneity in the population need not be an issue for the empirical application of the

Ising model to cross-sectional data, and illustrate an alternative interpretation that allows for heterogeneity with respect to the model's statistical relations.

It is well-known that statistical relations that are found at the group level may be fundamentally different from the relations that are found at the participant level (Kievit, Frankenhuis, Waldorp, & Borsboom, 2013) —a phenomenon that is commonly referred to as Simpson's paradox (Simpson, 1951) or the ecological fallacy (Robinson, 1950)— which is one of the factors that has led to the division of psychological science in idiographic and cross-sectional approaches (Cronbach, 1957; Molenaar, 2004). This division can also be observed in the network psychometric literature (e.g., E. Bos & Wanders, 2016; F. M. Bos, Snippe, Bruggeman, Wichers, & van der Krieke, in press; Fisher, 2015; Fisher, Medaglia, & Jeronimus, 2018). But if statistical relations can vary at the individual level, how can idiographic networks then be related to Ising models that are estimated from cross-sectional data?

The answer is given by Marsman (2019), who used the characterization in Eq. (3) to draw two broad conclusions. First, the characterization in Eq. (3) using latent topologies illustrates how one can retrieve an Ising model when the network's topology is a random effect that may vary at an individual level. That is, idiographic networks that embody a unique set of statistical relations between variables at the participant level may very well be consistent with an Ising model at the population level. This underscores our point that homogeneity is not a necessary condition for retrieving an Ising model from cross-sectional data, but is merely a sufficient condition. Second, there exists a concrete, formal relation between the cross-sectional associations and idiographic relations. In the theory of Fortuin and Kasteleyn, the associations σ_{ij} that are estimated from cross-sectional data are monotonically related to the proportion θ_{ij} of idiographic topological structures that have a connection between nodes i and j —i.e., Eq. (5). Such a definite connection between cross-sectional models and idiographic models is rather unique, in particular in the network psychometric literature, and shows how group-level phenomena can be generated from idiographic models (e.g., Savi et al., 2019). Moreover, Marsman (2019) illustrates that the group-level random-cluster model can be used as prior information to learn about the idiographic topologies, which, in turn, can be used to learn about the underlying population model. In sum, Fortuin and Kasteleyn's theory offers a concrete counterexample to the idea that a homogeneous population is necessary for recovering an Ising model cross-sectionally, and it also forms an essential bridge between the idiographic and cross-sectional approaches in network psychometrics.

Subgroup Heterogeneity

Our idiographic formulation of the Ising model does not explain all forms of heterogeneity, and BSHDW offer a concrete example that lies at the boundary of what our approach can cover; having distinct subpopulations. Heterogeneity of groups poses an interesting challenge for the models that we discussed. In principle, an encompassing network model could reflect this form of heterogeneity by, for example, stipulating a mixture of random graph models in Eq. (3). The node marginal in Eq. (3) is probably not a particular Ising model if the edge-marginal is a mixture. This is because the parameter mapping in Eq. (5) cannot cover the mixture specification. But, Eq. (4) does ensure that there is a link between the connectivity in idiographic networks and cross-sectional correlations:

$$\sum_{g=1}^{n_g} p(X_i = x_j \mid G = g)p(G = g) = \frac{1}{2} + \frac{1}{2}\sum_{g=1}^{n_g} p(X_i \leftrightarrow x_j \mid G = g)p(G = g),$$

for groups $g = 1, \ldots, n_g$. In principle, such an idea might work if one can efficiently regularize its parameters, otherwise it can become rather unwieldy and difficult to estimate. Eqs. (3) and (4) help ensure that we can check the fit of the encompassing mixture model via its node-marginal.

However, what is unclear is whether a mixture approach would improve the fit of the marginal distribution to the data at hand. The fact that we do not know whether BSHDW's proposal improves the fit in their empirical example highlights a concern that we have with their empirical assessments: BSHDW appear to criticize the fit of the model to the data at hand without analyzing it. In our opinion, the lack of statistical validation could undermine their analysis, and jeopardize their conclusions. The recovery of different clusters in BSHDW's empirical illustrations, for example, does not invalidate the fit of the Ising model. In fact, it is well-known that the Ising model can accommodate such clusters (e.g., Marsman et al., 2018; Marsman, Waldorp, & Maris, 2017). Marsman (2019) recently illustrated an approach that can used for assessing the empirical fit of the Ising model. It is based on the random effects specification in Eq. (3) in combination with recent plausible value theory. Specifically, Marsman, Maris, Bechger, and Glas (2016) showed that the cross-sectional distribution of draws from the posterior distributions of random effects diverges from the population model if the population model is misspecified. Thus, one can assess the fit of the node-marginal by inspecting the distribution of the posterior draws of the idiographic networks and comparing that to its population model, the random-cluster model.

Low-Rank Models Offer a Workable Alternative to Sparsity

A second concern that BSHDW voiced about the use of the Ising model in psychopathology, or psychology more generally, is that "*psychopathological networks may not be sparse*" (p. 8). To be clear, the assumption that the network is sparse is not related to the Ising model, but to the eLasso method that is used to estimate the Ising model. The eLasso however is an algorithm, not a model, and hence BSHDWs' argument is misdirected. Waldorp, Marsman, and Maris (2019), for example, show how the assumption of having a sparse connected network underlies the eLasso method, and what the effect of violating this assumption is on both parameter estimation and the prediction of node states. Specifically, it was shown that the estimation error increases when the underlying network is no longer sparse, even though prediction accuracy actually improves. Thus for recovering the true underlying graph structure using the eLasso method it is critical to have a truly sparse graph or have sufficient data to overcome this prior assumption (Epskamp, Kruis, & Marsman, 2017). If one expects the underlying graph to be densely connected, instead of sparsely connected, one should take a different approach, such as the low-rank approach detailed in Marsman et al. (2015).

The low-rank Ising model was proposed by Marsman et al. (2015), and applies the Eckart and Young (1936) Theorem to formulate a low-rank approximation to the Ising model's association matrix,

$$\mathbf{\Sigma} \approx \sum_{r=1}^{R} \lambda_r \, \mathbf{q}_r \mathbf{q}_r^{\mathsf{T}},$$

in which all but the largest R eigenvalues λ_r are equated to zero (Marsman et al., 2018, 2015). We were pleasantly surprised that BSHDW proposed "*...a singular value decomposition of* $\mathbf{X}$*, whereby a low-rank approximation based on eigenvectors from* $\mathbf{X}\mathbf{X}'$ *or* $\mathbf{X}'\mathbf{X}$ *is obtained (Eckart & Young, 1936...*" (p. 15), although our surprise would have been more pleasant if a reference to the original proposal was included. We believe that there is much merit to this approach, although we would argue that the restriction should be imposed on the association matrix, and not the raw data, for reasons of statistical cleanliness.

Associations are never really excluded in the low-rank Ising model, which summarizes patterns between the associations of several variables in the eigenvectors $\mathbf{q}_r$ instead. In Marsman et al. (2015), for example, the eigenvectors are used to express contrasts between distinct topics in a large-scale educational survey, a set-up that generates correlational structures that are similar to that of multidimensional IRT. But where one can erroneously conclude that the underlying network is sparse when using the eLasso method, one can also erroneously conclude that the underlying network is dense when using the low-rank approach (Epskamp et al., 2017). To get the best of both worlds, one may consider a mix of the two approaches, and bridge between a sparse and dense network structure, such as the fused latent and graphical IRT approach to estimate the Ising model, originally proposed by Chen, Li, Liu, and Ying (2018). In sum, as we have shown here, several alternative approaches to eLasso exist in the network psychometric literature, some of which are closely related to BSHDW's proposed solutions.

Concluding Comments

In this commentary we have used the seminal work of Fortuin and Kasteleyn to refute BSHDWs' theses that homogeneity is a necessary condition for cross-sectional applications of the Ising model, and that graphical models have nothing to do with networks. Fortuin and Kasteleyn's unified approach towards random graph models and graphical models allowed us to reveal how BSHDWs' critiques and proposed solutions, which are heavily influenced by ideas from social network analysis, translate to the realm of network psychometrics. We discovered, for example, that two of BSHDW's proposed solutions are intimately related to existing approaches in the network psychometric literature, which underscores the value of their contribution. But BSHDW's contribution also highlights that the two frameworks span largely independent streams of literature. We hope that someday we will be able to construct a grand unified theory of the two statistical realms, and we believe that the encompassing network approach of Fortuin and Kasteleyn is an important first step in that direction.

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Appendix

The relation between Eqs. (1) and (3)

The original formulation of Eq. (3) by Fortuin and Kasteleyn is equal to

$$\sum_{\mathbf{w}} \frac{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\left\{\theta_{ij}\,\delta_{(w_{ij},1)}\,\delta_{(x_i,\,x_j)} + (1-\theta_{ij})\,\delta_{(w_{ij},0)}\right\}\exp\left(\sum_{i=1}^{p} x_i\mu_i\right)}{\sum_{\mathbf{x}}\sum_{\mathbf{w}}\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\left\{\theta_{ij}\,\delta_{(w_{ij},1)}\,\delta_{(x_i,\,x_j)} + (1-\theta_{ij})\,\delta_{(w_{ij},0)}\right\}\exp\left(\sum_{i=1}^{p} x_i\mu_i\right)}, \tag{6}$$

where $\delta_{(a,\,b)}$ is an indicator function that is equal to one whenever $a = b$ and zero otherwise. We first illustrate that Eqs. (3) and (6) are, in fact, the same, and characterize the marginal $p(\mathbf{w})$ and conditional $p(\mathbf{x} \mid \mathbf{w})$. Then, we illustrate the relation between Eqs. (1) and (6).

We start with formulating the marginal distribution $p(\mathbf{w})$. Following Grimmet (2006), we observe that the product can be reformulated as

$$\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\left\{\theta_{ij}\,\delta_{(w_{ij},1)}\,\delta_{(x_i,\,x_j)} + (1-\theta_{ij})\,\delta_{(w_{ij},0)}\right\} = \prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\,\mathbf{1}_F(\mathbf{w},\,\mathbf{x}),$$

where $\mathbf{1}_F(\mathbf{w},\,\mathbf{x})$ is the indicator function that $\delta_{(x_i,\,x_j)} = 1$ whenever $\delta_{(w_{ij},1)} = 1$, i.e., an indicator function for the event

$$F = \{(\mathbf{x},\,\mathbf{w}) : \delta_{(x_i,\,x_j)} = 1 \text{ whenever } \delta_{(w_{ij},\,1)} = 1\}.$$

Observe that $\mathbf{1}_F(\mathbf{w},\,\mathbf{x}) = 1$ only when $\mathbf{x}$ is constant on every cluster of $\mathbf{w}$. But in that case

$$\prod_{i=1}^{p}\exp\left(x_i\mu_i\right)\mathbf{1}_F(\mathbf{w},\,\mathbf{x}) = \prod_{c=1}^{\kappa(\mathbf{w})}\exp\left(\sum_{i\in V_c} x_i\mu_i\right)\mathbf{1}_F(\mathbf{w},\,\mathbf{x}) \Rightarrow \prod_{c=1}^{\kappa(\mathbf{w})}\exp\left(x_c\sum_{i\in V_c}\mu_i\right),$$

where $\kappa(\mathbf{w})$ denotes the number of open clusters, V_c is the set of nodes in the c-th cluster, and x_c is the state of the nodes in the c-th cluster. We can now express the marginal model $p(\mathbf{w})$ as

$$\begin{aligned} p(\mathbf{W} = \mathbf{w}) &= \frac{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\ \sum_{\mathbf{x}}\prod_{c=1}^{\kappa(\mathbf{w})}\exp\left(\sum_{i\in V_c} x_i\mu_i\right)\mathbf{1}_F(\mathbf{w},\,\mathbf{x})}{\sum_{\mathbf{w}}\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\ \sum_{\mathbf{x}}\prod_{c=1}^{\kappa(\mathbf{w})}\exp\left(\sum_{i\in V_c} x_i\mu_i\right)\mathbf{1}_F(\mathbf{w},\,\mathbf{x})} \\ &= \frac{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\prod_{c=1}^{\kappa(\mathbf{w})}\beta_c}{\sum_{\mathbf{w}}\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\prod_{c=1}^{\kappa(\mathbf{w})}\beta_c}, \end{aligned}$$

where we recognize the cluster-weight $\beta_c = 2\cosh\left(\sum_{i\in V_c}\mu_i\right)$ of Cioletti and Vila (2016). Given the expression for the marginal distribution, it is now a trivial exercise to find the conditional distribution

$$
\begin{aligned}
p(\mathbf{x} \mid \mathbf{w}) &= \frac{\frac{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\ \prod_{c=1}^{\kappa(\mathbf{w})}\exp\left(\sum_{i\in V_c}x_i\mu_i\right)}{\sum_{\mathbf{w}}\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\prod_{c=1}^{\kappa(\mathbf{w})}\beta_c}}{\frac{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\prod_{c=1}^{\kappa(\mathbf{w})}\beta_c}{\sum_{\mathbf{w}}\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\prod_{c=1}^{\kappa(\mathbf{w})}\beta_c}} \\
&= \frac{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\ \prod_{c=1}^{\kappa(\mathbf{w})}\exp\left(\sum_{i\in V_c}x_i\mu_i\right)}{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\theta_{ij}^{w_{ij}}(1-\theta_{ij})^{1-w_{ij}}\prod_{c=1}^{\kappa(\mathbf{w})}\beta_c} \\
&= \prod_{c=1}^{\kappa(\mathbf{w})}\frac{\exp\left(\sum_{i\in V_c}x_i\mu_i\right)}{\beta_c} \\
&= \prod_{c=1}^{\kappa(\mathbf{w})}\frac{\exp\left(\sum_{i\in V_c}x_i\mu_i\right)}{\exp\left(\sum_{i\in V_c}\mu_i\right)+\exp\left(-\sum_{i\in V_c}\mu_i\right)},
\end{aligned}
$$

This confirms that Eqs. (3) and (6) are indeed equivalent, and we can now use the characterization (6) to work out the relation to the Ising model in Eq. (1).

We wish to show that the marginal distribution $p(\mathbf{x})$ of Eq. (6) gives the Ising model. To this aim, we start with summing over the edge configurations $\mathbf{w}$, and readily find

$$
p(\mathbf{X}=\mathbf{x}) = \frac{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\left\{\theta_{ij}\delta_{(x_i,x_j)}+1-\theta_{ij}\right\}\exp\left(\sum_{i=1}^{p}\mu_i x_i\right)}{\sum_{\mathbf{x}}\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\left\{\theta_{ij}\delta_{(x_i,x_j)}+1-\theta_{ij}\right\}\exp\left(\sum_{i=1}^{p}\mu_i x_i\right)}.
$$

Next, we plug in the relation $\theta_{ij} = 1-\exp(-2\sigma_{ij})$ from Eq. (5) to obtain

$$
\begin{aligned}
p(\mathbf{X}=\mathbf{x}) &= \frac{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\left\{(1-\exp(-2\sigma_{ij}))\delta_{(x_i,x_j)}+\exp(-2\sigma_{ij})\right\}\exp\left(\sum_{i=1}^{p}\mu_i x_i\right)}{\sum_{\mathbf{x}}\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\left\{(1-\exp(-2\sigma_{ij}))\delta_{(x_i,x_j)}+\exp(-2\sigma_{ij})\right\}\exp\left(\sum_{i=1}^{p}\mu_i x_i\right)} \\
&= \frac{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\exp\left(2\sigma_{ij}\delta_{(x_i,\,x_j)}\right)\exp\left(\sum_{i=1}^{p}\mu_i x_i\right)}{\sum_{\mathbf{x}}\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\exp\left(2\sigma_{ij}\delta_{(x_i,\,x_j)}\right)\exp\left(\sum_{i=1}^{p}\mu_i x_i\right)},
\end{aligned}
$$

and since $\delta_{(x_i,\,x_j)} = \frac{1}{2}(1+x_i x_j)$, we have, in fact, obtained the Ising model

$$
\begin{aligned}
p(\mathbf{X}=\mathbf{x}) &= \frac{\exp\left(\sum_{i=1}^{p}\mu_i x_i+\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\sigma_{ij}x_i x_j+\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\sigma_{ij}\right)}{\sum_{\mathbf{x}}\exp\left(\sum_{i=1}^{p}\mu_i x_i+\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\sigma_{ij}x_i x_j+\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\sigma_{ij}\right)} \\
&= \frac{\exp\left(\sum_{i=1}^{p}\mu_i x_i+\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\sigma_{ij}x_i x_j\right)}{\sum_{\mathbf{x}}\exp\left(\sum_{i=1}^{p}\mu_i x_i+\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\sigma_{ij}x_i x_j\right)}.
\end{aligned}
$$